Fall 2016

61DM Handout: Inclusion-Exclusion Principle

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We begin with the binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

The binomial theorem follows from considering the coefficient of $x^k y^{n-k}$, which is the number of ways of choosing x from k of the n terms in the product and y from the remaining $n-k$ terms, and is thus $\binom{n}{k}$. One can also prove the binomial theorem by induction on n using Pascal's identity. The binomial theorem is a useful fact. For example, we can use the binomial theorem with $x=-1$ and $y=1$ to obtain

$$0 = (1-1)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n}.$$

Thus, the even binomial coefficients add up to the odd coefficients for $n \geq 1$.

The inclusion-exclusion principle is an important tool in counting.

Note that if we have two finite sets A_1 and A_2, then

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|. \tag{1}$$

This is because every element is either not in A_1 nor in A_2, in A_1 but not in A_2, in A_2 but not in A_1, or in $A_1 \cup A_2$. In each of the four cases, they are counted the same number of times on the left and right side of the equation, giving the equality.

We can iteratively apply Equation (1). For example, the next case says that

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|. \tag{2}$$

We obtain this by substituting in $A_1 \cup A_2$ and A_3 into Equation (1), and then apply Equation (1) two more times:

$$\begin{aligned}
|(A_1 \cup A_2) \cup A_3| &= |A_1 \cup A_2| + |A_3| - |(A_1 \cup A_2) \cap A_3| \\
&= |A_1 \cup A_2| + |A_3| - |(A_1 \cap A_3) \cup (A_2 \cap A_3)| \\
&= (|A_1| + |A_2| - |A_1 \cap A_2|) + |A_3| - (|A_1 \cap A_3| + |A_2 \cap A_3| - |(A_1 \cap A_3) \cap (A_2 \cap A_3)|) \\
&= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.
\end{aligned}$$

Generalizing, we can use induction on n to obtain the inclusion-exclusion principle:

$$
\begin{aligned}
|\bigcup_{i=1}^{n} A_i| &= \sum_{i=1}^{n}|A_i| - \sum_{1\le i<j\le n}|A_i\cap A_j| + \sum_{1\le i<j<k\le n}|A_i\cap A_j\cap A_k| - \cdots - (-1)^n|\bigcap_{i=1}^{n} A_i| \\
&= \sum_{S\subset[n],S\neq\emptyset}(-1)^{|S|+1}|\bigcap_{i\in S} A_i|.
\end{aligned}
$$

Another way to obtain the inclusion-exclusion principle is to notice that each element x contributes the same number to each side of the equation. Suppose $S \subset [n]$ is the set of i for which $x \in A_i$. If S is empty, so that x is none of the A_i, then x contributes 0 to both sides. Otherwise, x contributes 1 to the left hand side, and $k - \binom{k}{2} + \binom{k}{3} - \cdots - (-1)^k\binom{k}{k} = \sum_{i=1}^{k}(-1)^{i+1}\binom{k}{i} = 1$ by the consequence of the binomial theorem discussed above.

A fixed point of a function $f : X \to X$ which maps a set X to itself is an element x such that $f(x) = x$. A permutation $\pi : [n] \to [n]$ with no fixed point is known as a derangement. We can count the number D_n of derangements of $[n]$ using the inclusion-exclusion principle. Let A_i be the set of permutations π of $[n]$ with $\pi(i) = i$, i.e., with i as a fixed point. Then $\bigcup_{i=0}^{n} A_i$ is the set of permutations of $[n]$ with at least one fixed point, and so $n! - |\bigcup_{i=0}^{n} A_i|$ is the number D_n of derangements of $[n]$. By the inclusion-exclusion principle, we have

$$\bigcup_{i=0}^{n} A_i = \sum_{S\subset[n],S\neq\emptyset}(-1)^{|S|+1}|\bigcap_{i\in S} A_i|.$$

Note that $\bigcap_{i\in S} A_i$ is the set of permutations of $[n]$ with each $i \in S$ mapping to itself. There are $(n-|S|)!$ such permutations as they just permute the $n-|S|$ elements of $[n]\setminus S$. Also, the number of S with $|S| = k$ is $\binom{n}{k}$. We have $\binom{n}{k}(n-k)! = \frac{n!}{k!(n-k)!}(n-k)! = n!/k!$. Hence,

$$\left|\bigcup_{i=0}^{n} A_i\right| = \sum_{k=1}^{n}(-1)^{k+1}n!/k!.$$

and

$$D_n = n! - \bigcup_{i=0}^{n} A_i = n! - \sum_{k=1}^{n}(-1)^{k+1}n!/k! = \sum_{k=0}^{n}(-1)^k n!/k! = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}.$$

You may remember from calculus that $\sum_{k=0}^{n}\frac{z^k}{k!}$ is the Taylor series approximation for e^z. Substituting in $z = -1$, the number of derangements of $[n]$ is very close to $n!/e$. Hence, the probability that a random permutation is a derangement is very close to $1/e$.